

# The DuBois–Reymond Fundamental Lemma of the Fractional Calculus of Variations and an Euler–Lagrange Equation Involving only Derivatives of Caputo

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## Abstract

Derivatives and integrals of non-integer order were introduced more than three centuries ago, but only recently gained more attention due to their application on nonlocal phenomena. In this context, the Caputo derivatives are the most popular approach to fractional calculus among physicists, since differential equations involving Caputo derivatives require regular boundary conditions. Motivated by several applications in physics and other sciences, the fractional calculus of variations is currently in fast development. However, all current formulations for the fractional variational calculus fail to give an Euler–Lagrange equation with only Caputo derivatives. In this work, we propose a new approach to the fractional calculus of variations by generalizing the DuBois–Reymond lemma and showing how Euler–Lagrange equations involving only Caputo derivatives can be obtained.

**Keywords:** fractional calculus, fractional calculus of variations, DuBois–Reymond lemma, Euler–Lagrange equations in integral and differential forms.

**Mathematics Subject Classification 2010:** 49K05, 26A33, 34A08.

## 1 Introduction

The fractional calculus with derivatives and integrals of non-integer order started more than three centuries ago, with l'Hôpital and Leibniz, when the derivative of order  $1/2$  was suggested [1]. This subject was then considered by several mathematicians like Euler, Fourier, Liouville, Grunwald, Letnikov, Riemann, and many others up to nowadays. Although the fractional calculus is almost as old as the usual integer order calculus, only in the last three decades it has gained more attention due to its many applications in various fields of science, engineering, economics, biomechanics, etc. (see [2–6] for a review).

Fractional derivatives are nonlocal operators and are historically applied in the study of nonlocal or time dependent processes. The first and well established application of fractional calculus in physics was in the framework of anomalous diffusion, which is related to features observed in many physical systems, e.g., in dispersive transport in amorphous semiconductor, liquid crystals, polymers, proteins, etc. [7,8]. Recently, the study of nonlocal quantum phenomena through fractional calculus began a fast development, where the nonlocal effects are due to either long-range interactions or time-dependent processes with many scales [4,9–12]. Relativistic quantum mechanics [13–15], field theories [16–18], and gravitation [19], have been considered in the context of fractional calculus. The physical and geometrical meaning of the fractional derivatives has been investigated by several authors [20,21]. It has been shown that the Stieltjes integral can be

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interpreted as the real distance passed by a moving object [20, 21]. The most popular operators of fractional calculus, namely the Riemann–Liouville [1] and Caputo [22, 23] fractional operators, are then given similar physical interpretations [20, 21].

One of the most remarkable applications of fractional calculus in physics is in the context of classical mechanics. Riewe showed that a Lagrangian involving fractional time derivatives leads to an equation of motion with nonconservative forces, such as friction [24, 25]. It is a striking result since frictional and nonconservative forces are beyond the usual macroscopic variational treatment and, consequently, beyond the most advanced methods of classical mechanics [26]. Riewe generalized the usual variational calculus to Lagrangians depending on fractional derivatives, in order to deal with nonconservative forces [24, 25]. Recently, several approaches have been developed in order to generalize the least action principle and the Euler–Lagrange equations to include fractional derivatives [24, 25, 27–34]. In this new formalism, because of the nonlocal properties of fractional time derivatives, the Euler–Lagrange equations appear to not respect the causality principle. This difficulty is under investigation by several authors [35, 36]. On the other hand, in all the above mentioned references, the Euler–Lagrange equations always depend on the Riemann–Liouville or mixed Caputo and Riemann–Liouville derivatives. For the state of the art of the fractional calculus of variations and respective fractional Euler–Lagrange equations, we refer the reader to the recent book [37].

It is important to remark that while the Riemann–Liouville fractional derivatives [1] are historically the most studied approach to fractional calculus, the Caputo [22, 23] approach to fractional derivatives is the most popular among physicists and scientists, because the differential equations defined in terms of Caputo derivatives require regular initial and boundary conditions. Furthermore, differential equations with Riemann–Liouville derivatives require non-standard fractional initial and boundary conditions that lead, in general, to singular solutions, thus limiting their application in physics and science [6]. On the other hand, within current formulations of the fractional calculus of variations, even Lagrangians depending only on Caputo derivatives lead to Euler–Lagrange equations with Riemann–Liouville derivatives (see [38] and references therein). This is a consequence of the Lagrange method to extremize functionals: application of integration by parts for Caputo derivatives in the Gateaux derivative of the functional, relates Caputo to Riemann–Liouville derivatives. In the present work, we propose a different approach, by generalizing the DuBois–Reymond fundamental lemma to variational functionals with Caputo derivatives. This new result enable us to obtain an Euler–Lagrange equation in integral form, containing only Caputo derivatives. Moreover, when the Lagrangian is  $C^2$  in its domain, we obtain an Euler–Lagrange fractional differential equation depending only on Caputo derivatives by taking the Caputo derivative on both sides of the Euler–Lagrange equation in integral form.

The article is organized as follows. In Section 2, we review the basic notions of Riemann–Liouville and Caputo fractional calculus, that are needed to formulate the fractional problem of the calculus of variations. The fractional DuBois–Reymond lemma is proved in Section 3, while in Section 4 we obtain the Euler–Lagrange equations in integral and differential forms. Finally, Section 5 presents the main conclusions of our work. Although we adopt here the Caputo fractional calculus, the results can be straightforward rewritten within other approaches, like the Riemann–Liouville one.

## 2 The Fractional Calculus

There are several definitions of fractional order derivatives. These definitions include the Riemann–Liouville, Caputo, Riesz, Weyl, and Grunwald–Letnikov operators [1–5, 39, 40]. In this Section, we review some definitions and properties of the Caputo and Riemann–Liouville fractional calculus. Although our first main objective is to obtain a DuBois–Reymond lemma for functionals with Caputo fractional derivatives, the Riemann–Liouville derivatives are also defined here because they naturally arise in the formulation.

Despite many different approaches to fractional calculus, several known formulations are connected with the analytical continuation of the Cauchy formula for  $n$ -fold integration.

**Theorem 2.1** (Cauchy formula). *Let  $f : [a, b] \rightarrow \mathbb{R}$  be Riemann integrable in  $[a, b]$ . The  $n$ -fold integration of  $f$ ,  $n \in \mathbb{N}$ , is given by*

$$\begin{aligned} \int_a^x f(\tilde{x})(d\tilde{x})^n &= \int_a^x \int_a^{x_n} \int_a^{x_{n-1}} \cdots \int_a^{x_3} \int_a^{x_2} f(x_1) dx_1 dx_2 \cdots dx_{n-1} dx_n \\ &= \frac{1}{\Gamma(n)} \int_a^x \frac{f(u)}{(x-u)^{1-n}} du, \end{aligned} \quad (1)$$

where  $\Gamma$  is the Euler gamma function.

The proof of Cauchy's formula (1) can be found in several textbooks, for example, it can be found in [1]. The analytical continuation of (1) gives a definition for integration of non integer (or fractional) order. This fractional order integration is the building block of the Riemann–Liouville and Caputo calculus, the two most popular formulations of fractional calculus, as well as several other approaches [1–5, 39, 40]. The fractional integrations obtained from (1) are historically called Riemann–Liouville fractional integrals.

**Definition 2.1** (Left and right Riemann–Liouville fractional integrals). *Let  $\alpha \in \mathbb{R}_+$ . The operators  ${}_a J_x^\alpha$  and  ${}_x J_b^\alpha$ , defined on  $L_1([a, b])$  by*

$${}_a J_x^\alpha f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(u)}{(x-u)^{1-\alpha}} du \quad (2)$$

and

$${}_x J_b^\alpha f(x) := \frac{1}{\Gamma(\alpha)} \int_x^b \frac{f(u)}{(u-x)^{1-\alpha}} du, \quad (3)$$

where  $a, b \in \mathbb{R}$  with  $a < b$ , are called the left and the right Riemann–Liouville fractional integrals of order  $\alpha$ , respectively.

For an integer  $\alpha$ , the fractional Riemann–Liouville integrals (2) and (3) coincide with the usual integer order  $n$ -fold integration (1). Moreover, from definitions (2) and (3), it is easy to see that the Riemann–Liouville fractional integrals converge for any integrable function  $f$  if  $\alpha > 1$ . Furthermore, it is possible to prove the convergence of (2) and (3) even when  $0 < \alpha < 1$  [40].

The integration operators  ${}_a J_x^\alpha$  and  ${}_x J_b^\alpha$  play a fundamental role in the definition of Caputo and Riemann–Liouville fractional derivatives. In order to define the Riemann–Liouville derivatives, we recall that, for positive integers  $n > m$ , it follows the identity  $D_x^m f(x) = D_x^n J_x^{n-m} f(x)$ , where  $D_x^m$  is the ordinary derivative  $d^m/dx^m$  of order  $m$ .

**Definition 2.2** (Left and right Riemann–Liouville fractional derivatives). *The left and the right Riemann–Liouville fractional derivatives of order  $\alpha \in \mathbb{R}_+$  are defined, respectively, by*

$${}_a D_x^\alpha f(x) := D_x^n {}_a J_x^{n-\alpha} f(x)$$

and

$${}_x D_b^\alpha f(x) := (-1)^n D_{xx}^n {}_x J_b^{n-\alpha} f(x)$$

with  $n = [\alpha] + 1$ ; that is,

$${}_a D_x^\alpha f(x) := \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x \frac{f(u)}{(x-u)^{1+\alpha-n}} du \quad (4)$$

and

$${}_x D_b^\alpha f(x) := \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_x^b \frac{f(u)}{(u-x)^{1+\alpha-n}} du. \quad (5)$$

The Caputo fractional derivatives are defined in a similar way as the Riemann–Liouville ones, but exchanging the order between integration and differentiation.

**Definition 2.3** (Left and right Caputo fractional derivatives). *The left and the right Caputo fractional derivatives of order  $\alpha \in \mathbb{R}_+$  are defined, respectively, by  ${}_a^C D_x^\alpha f(x) := {}_a J_x^{n-\alpha} D_x^n f(x)$  and  ${}_x^C D_b^\alpha f(x) := (-1)_x^n J_b^{n-\alpha} D_x^n f(x)$  with  $n = [\alpha] + 1$ ; that is,*

$${}_a^C D_x^\alpha f(x) := \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{f^{(n)}(u)}{(x-u)^{1+\alpha-n}} du \quad (6)$$

and

$${}_x^C D_b^\alpha f(x) := \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^b \frac{f^{(n)}(u)}{(u-x)^{1+\alpha-n}} du, \quad (7)$$

where  $a \leq x \leq b$  and  $f^{(n)}(u) = \frac{d^n f(u)}{du^n} \in L_1([a, b])$  is the ordinary derivative of integer order  $n$ .

An important consequence of definitions (4)–(7) is that the Riemann–Liouville and Caputo fractional derivatives are nonlocal operators. The left (right) differ-integration operator (4) and (6) ((5) and (7)) depend on the values of the function at left (right) of  $x$ , i.e.,  $a \leq u \leq x$  ( $x \leq u \leq b$ ).

**Remark 2.1.** *When  $\alpha$  is an integer, the Riemann–Liouville fractional derivatives (4) and (5) reduce to ordinary derivatives of order  $\alpha$ . On the other hand, in that case, the Caputo derivatives (6) and (7) differ from integer order ones by a polynomial of order  $\alpha - 1$  [3, 40].*

**Remark 2.2.** *For  $0 < \alpha < 1$ , and because they are given by a first order derivative of a fractional integral, the Riemann–Liouville fractional derivatives (4) and (5) can be applied to non-differentiable functions. One can even take Riemann–Liouville derivatives of nowhere differentiable functions, like the Weierstrass function [41]. In fact, one of the main reasons for using fractional calculus is the possibility to study non-differentiability [42, 43].*

**Remark 2.3.** *It is important to note that, while the Caputo derivatives (6) and (7) of a constant are zero for all  $\alpha > 0$ ,  ${}_a^C D_x^\alpha 1 = {}_x^C D_b^\alpha 1 = 0$ , the Riemann–Liouville derivatives (4) and (5) are not zero for  $\alpha \notin \mathbb{N}$ :*

$${}_a D_x^\alpha 1 = \frac{(x-a)^{-\alpha}}{\Gamma(1-\alpha)}, \quad {}_x D_b^\alpha 1 = \frac{(b-x)^{-\alpha}}{\Gamma(1-\alpha)}. \quad (8)$$

For  $\alpha \in \mathbb{N}$ , the right hand sides of (8) become equal to zero due to the poles of the gamma function. Furthermore, for the power function  $(x-a)^\beta$  one has

$${}_a D_x^\alpha (x-a)^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} (x-a)^{\beta-\alpha}.$$

**Remark 2.4.** *Let  $\alpha \in ]0, 1[$ . If the Riemann–Liouville and the Caputo fractional derivatives of order  $\alpha$  exist, then they are connected with each other by the following relations:*

$${}_a^C D_x^\alpha f(x) = {}_a D_x^\alpha f(x) - \frac{f(a)}{\Gamma(1-\alpha)} (x-a)^{-\alpha}$$

and

$${}_x^C D_b^\alpha f(x) = {}_x D_b^\alpha f(x) - \frac{f(b)}{\Gamma(1-\alpha)} (b-x)^{-\alpha}.$$

Therefore, if  $f(a) = 0$  ( $f(b) = 0$ ), then  ${}_a^C D_x^\alpha f(x) = {}_a D_x^\alpha f(x)$  ( ${}_x^C D_b^\alpha f(x) = {}_x D_b^\alpha f(x)$ ) [3, 40].

We make use of the following three properties of the Riemann–Liouville and Caputo derivatives (Theorems 2.2, 2.3 and 2.4) in the proofs of our results.

**Theorem 2.2** (see, e.g., [3, 40]). *Let  $\alpha > 0$ . For every  $f \in L_1([a, b])$  one has*

$${}_a D_x^\alpha {}_a J_x^\alpha f(x) = f(x), \quad {}_x D_b^\alpha {}_x J_b^\alpha f(x) = f(x) \quad (9)$$

and

$${}_a^C D_x^\alpha {}_a J_x^\alpha f(x) = f(x), \quad {}_x^C D_b^\alpha {}_x J_b^\alpha f(x) = f(x), \quad (10)$$

almost everywhere.

**Theorem 2.3** (Fundamental theorem of Caputo calculus — see, e.g., [3, 40]). *Let  $0 < \alpha < 1$  and  $f$  be a differentiable function in  $[a, b]$ . The following two equalities hold:*

$${}_a J_b^{\alpha C} D_x^\alpha f(x) = f(b) - f(a) \quad (11)$$

and

$${}_b J_a^{\alpha C} D_b^\alpha f(x) = f(a) - f(b). \quad (12)$$

**Theorem 2.4** (Integration by parts — see, e.g., [39]). *Let  $0 < \alpha < 1$  and  $f$  be a differentiable function in  $[a, b]$ . For any function  $g \in L_1([a, b])$  one has*

$$\int_a^b g(x) {}_a^C D_x^\alpha f(x) dx = \int_a^b f(x) {}_x D_b^\alpha g(x) dx + [{}_x D_b^{\alpha-1} g(x) f(x)]_a^b \quad (13)$$

and

$$\int_a^b g(x) {}_x D_b^\alpha f(x) dx = \int_a^b f(x) {}_a D_x^\alpha g(x) dx + [(-1)^n {}_a D_x^{\alpha-1} g(x) f(x)]_a^b. \quad (14)$$

The proof of relations (9)–(12) can be found in several textbooks — see, e.g., [3, 40]. Relation (9) is a direct consequence of (4) and (5). Theorem 2.3 is the generalization of the fundamental theorem of calculus to the Caputo fractional calculus. It is important to mention that (11) and (12) do not hold in the Riemann–Liouville approach. Finally, we remark that the formulas of integration by parts (13) and (14) relate Caputo left (right) derivatives to Riemann–Liouville right (left) derivatives.

### 3 The Fractional DuBois–Reymond Lemma

The fundamental problem of the fractional calculus of variations consists to find a function  $y$  that maximizes or minimizes a given functional  $J$  [37]. We consider a functional with a Lagrangian depending on the independent variable  $x$ , function  $y$  and its left Caputo fractional derivative of order  $0 < \alpha < 1$ ,

$$J[y] = \int_a^b L\left(x, y, {}_a^C D_x^\alpha y\right) dx = \Gamma(\alpha) {}_a J_b^\alpha \left[(b-x)^{1-\alpha} L\left(x, y, {}_a^C D_x^\alpha y\right)\right], \quad (15)$$

where  ${}_a J_b^\alpha$  is seen as the left Riemann–Liouville fractional integral at  $x = b$ , subject to the boundary conditions  $y(a) = y_a$  and  $y(b) = y_b$ ,  $y_a, y_b \in \mathbb{R}$ . For  $\alpha = 1$ ,  $J$  is a functional of the classical calculus of variations [44]. In order to obtain a necessary condition for the extremum of (15), we use the following two lemmas.

**Lemma 3.1.** *Let  $f \in L_1([a, b])$  and  $0 < \alpha < 1$ . If there is a number  $\varepsilon \in ]a, b]$  such that*

$$|f(x)| \leq c(x-a)^\beta$$

*for all  $x \in [a, \varepsilon]$  with  $c > 0$  and  $\beta > -\alpha$ , then*

$$\lim_{x \rightarrow a^+} {}_a J_x^\alpha f(x) = 0. \quad (16)$$

**Proof.** From (2) one has

$$\left| \int_a^x \frac{f(u)}{(x-u)^{1-\alpha}} du \right| \leq c \int_a^x \frac{(u-a)^\beta}{(x-u)^{1-\alpha}} du = c \frac{\Gamma(\alpha)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} (x-a)^{\alpha+\beta}$$

for all  $x \in [a, \varepsilon]$ . Taking into account that  $\alpha + \beta > 0$ , it follows that

$$\left| \lim_{x \rightarrow a^+} \int_a^x \frac{f(u)}{(x-u)^{1-\alpha}} du \right| \leq c \frac{\Gamma(\alpha)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} \lim_{x \rightarrow a^+} (x-a)^{\alpha+\beta} = 0.$$

□

**Remark 3.1.** If  $0 < \beta \leq 1$ , then function  $f$  of Lemma 3.1 is a Hölder function of order  $\beta$ . If  $\beta = 0$ , then  $f$  is a simple bounded function.

**Lemma 3.2** (The DuBois–Reymond fundamental lemma of the fractional calculus of variations). *Let  $g$  be a differentiable function in  $[a, b]$  with  $g(a) = g(b) = 0$ , and let  $f \in L_1([a, b])$  be such that there is a number  $\varepsilon \in ]a, b]$  with  $|f(x)| \leq c(x-a)^\beta$  for all  $x \in [a, \varepsilon]$ , where  $c > 0$  and  $\beta > -\alpha$  are constants. Then,*

$${}_a J_b^\alpha \left( f(x) {}_a^C D_x^\alpha g(x) \right) = 0 \Rightarrow f \equiv K,$$

where  $K$  is a constant.

**Proof.** For any constant  $K$  we have

$$\begin{aligned} {}_a J_b^\alpha \left( (f(x) - K) {}_a^C D_x^\alpha g(x) \right) &= {}_a J_b^\alpha \left( f(x) {}_a^C D_x^\alpha g(x) \right) - K {}_a J_b^\alpha {}_a^C D_x^\alpha g(x) \\ &= -K {}_a J_b^\alpha {}_a^C D_x^\alpha g(x) \\ &= -K (g(b) - g(a)) \\ &= 0, \end{aligned} \tag{17}$$

where we used the fundamental theorem of Caputo calculus (11) and the hypothesis  $g(a) = g(b)$ . Let us choose

$$g(x) := {}_a J_x^\alpha (f(x) - K). \tag{18}$$

It can be seen that  $g(x)$ , defined by (18), is a differentiable function in  $[a, b]$ . Furthermore, from (16) of Lemma 3.1, we have  $g(a) = 0$  and, by choosing

$$K = \frac{1}{{}_a J_b^\alpha 1} {}_a J_b^\alpha f = \frac{\Gamma(\alpha + 1)}{(b-a)^\alpha} {}_a J_b^\alpha f,$$

we also have  $g(b) = 0$ . By inserting (18) into (17), and using (10), we get

$${}_a J_b^\alpha (f(x) - K) {}_a^C D_x^\alpha g(x) = {}_a J_b^\alpha (f(x) - K)^2 = \frac{1}{\Gamma(\alpha)} \int_a^b \frac{(f(u) - K)^2}{(b-u)^{1-\alpha}} du = 0.$$

Since  $b-x > 0$  for all  $x \in [a, b]$ , we conclude that  $f(x) = K$  for all  $x \in [a, b]$ .  $\square$

**Remark 3.2.** The fractional DuBois–Reymond lemma can be trivially formulated for Riemann–Liouville derivatives instead of Caputo ones. Indeed,  ${}_a^C D_x^\alpha g(x) = {}_a D_x^\alpha g(x)$  because  $g(a) = 0$ , as commented in Remark 2.4. Consequently, all results of the next section can also be formulated for functionals depending on Riemann–Liouville derivatives.

## 4 Fractional Euler–Lagrange Equations

The next Theorem gives a necessary condition for a function  $y$  to be an extremizer of the fractional variational functional defined by (15).

**Theorem 4.1** (The fractional Euler–Lagrange equation in integral form). *Let  $J$  be a functional of the form*

$$J[y] = \int_a^b L \left( x, y, {}_a^C D_x^\alpha y \right) dx = \Gamma(\alpha) {}_a J_b^\alpha \left[ (b-x)^{1-\alpha} L \left( x, y, {}_a^C D_x^\alpha y \right) \right],$$

defined in the class of functions  $y \in C^1([a, b])$  satisfying given boundary conditions  $y(a) = y_a$  and  $y(b) = y_b$ , and where  $L \in C^1([a, b] \times \mathbb{R}^2)$  is differentiable with respect to all of its arguments. If  $y$  is an extremizer of  $J$ , then  $y$  satisfies the following fractional Euler–Lagrange integral equation:

$${}_x J_b^\alpha \frac{\partial L(x, y, {}_a^C D_x^\alpha y)}{\partial y} + \frac{\partial L(x, y, {}_a^C D_x^\alpha y)}{\partial ({}_a^C D_x^\alpha y)} = \frac{K}{(b-x)^{1-\alpha}} \tag{19}$$

for all  $x \in [a, b]$ , where  $K$  is a constant.

**Proof.** Let  $y^*$  give an extremum to (15). We define a family of functions

$$y(x) = y^*(x) + \epsilon \eta(x), \quad (20)$$

where  $\epsilon$  is a constant and  $\eta \in C^1([a, b])$  is an arbitrary continuously differentiable function satisfying the boundary conditions  $\eta(a) = \eta(b) = 0$  (weak variations). From (20) and the boundary conditions  $\eta(a) = \eta(b) = 0$  and  $y^*(a) = y_a$ ,  $y^*(b) = y_b$ , it follows that function  $y$  is admissible:  $y \in C^1([a, b])$  with  $y(a) = y_a$  and  $y(b) = y_b$ . Let the Lagrangian  $L$  be  $C^1([a, b] \times \mathbb{R}^2)$ . Because  $y^*$  is an extremizer of functional  $J$ , the Gateaux derivative  $\delta J[y^*]$  needs to be identically null. For the functional (15),

$$\begin{aligned} \delta J[y^*] &= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left( \int_a^b L(x, y, {}^C D_x^\alpha y) dx - \int_a^b L(x, y^*, {}^C D_x^\alpha y^*) dx \right) \\ &= \int_a^b \left( \eta(x) \frac{\partial L(x, y^*, {}^C D_x^\alpha y^*)}{\partial y^*} + {}^C D_x^\alpha \eta(x) \frac{\partial L(x, y^*, {}^C D_x^\alpha y^*)}{\partial ({}^C D_x^\alpha y^*)} \right) dx \\ &= 0. \end{aligned} \quad (21)$$

By using (3) and relations (9) and (13), we get

$$\begin{aligned} \int_a^b \eta(x) \frac{\partial L(x, y^*, {}^C D_x^\alpha y^*)}{\partial y^*} dx &= \int_a^b \eta(x) {}_x D_b^\alpha J_b^\alpha \frac{\partial L(x, y^*, {}^C D_x^\alpha y^*)}{\partial y^*} dx \\ &= \int_a^b {}^C D_x^\alpha \eta(x) J_b^\alpha \frac{\partial L(x, y^*, {}^C D_x^\alpha y^*)}{\partial y^*} dx. \end{aligned} \quad (22)$$

Inserting (22) into (21), and using the definition (2) of left Riemann–Liouville fractional integral, we obtain for the first variation the following expression:

$$\begin{aligned} \delta J[y^*] &= \int_a^b {}^C D_x^\alpha \eta(x) \left( {}_x J_b^\alpha \frac{\partial L(x, y^*, {}^C D_x^\alpha y^*)}{\partial y^*} + \frac{\partial L(x, y^*, {}^C D_x^\alpha y^*)}{\partial ({}^C D_x^\alpha y^*)} \right) dx \\ &= \Gamma(\alpha) {}_a J_b^\alpha \left[ {}^C D_x^\alpha \eta(x) \left( {}_x J_b^\alpha \frac{\partial L(x, y^*, {}^C D_x^\alpha y^*)}{\partial y^*} + \frac{\partial L(x, y^*, {}^C D_x^\alpha y^*)}{\partial ({}^C D_x^\alpha y^*)} \right) (b-x)^{1-\alpha} \right] = 0. \end{aligned}$$

The fractional Euler–Lagrange equation (19) follows from Lemma 3.2. Note that the hypothesis  $|f(x)| \leq |c|(x-a)^\beta$  with  $\beta > -\alpha$  is satisfied because  $L \in C^1([a, b] \times \mathbb{R}^2)$  and  $y \in C^1([a, b])$ . Indeed, in our case  $f(x) = \left( {}_x J_b^\alpha \frac{\partial L(x, y^*, {}^C D_x^\alpha y^*)}{\partial y^*} + \frac{\partial L(x, y^*, {}^C D_x^\alpha y^*)}{\partial ({}^C D_x^\alpha y^*)} \right) (b-x)^{1-\alpha}$  tends to a number  $c$  when  $x \rightarrow a$ . Therefore,  $\beta = 0 > -\alpha$  and we are in conditions to apply Lemma 3.2.  $\square$

It is important to mention that in all previous approaches to the fractional calculus of variations, one eliminates the fractional derivative of function  $\eta$  appearing in the second integrand of (21) by performing an integration by parts [37]. However, from the integration by parts (13), this procedure gives an Euler–Lagrange fractional differential equation involving Riemann–Liouville derivatives (see [38] and references therein):

$$\frac{\partial L(x, y, {}^C D_x^\alpha y)}{\partial y} + {}_x D_b^\alpha \frac{\partial L(x, y, {}^C D_x^\alpha y)}{\partial ({}^C D_x^\alpha y)} = 0. \quad (23)$$

In contrast, in the proof of Theorem 4.1 we take a different procedure by computing an integration by parts on the first integrand instead of the second one. The new fractional DuBois–Reymond lemma takes then a prominent role.

One can obtain the Euler–Lagrange equation (23) from our result, by taking the right Riemann–Liouville derivative to both sides of (19):

$${}_x D_b^\alpha J_b^\alpha \frac{\partial L(x, y, {}^C D_x^\alpha y)}{\partial y} + {}_x D_b^\alpha \frac{\partial L(x, y, {}^C D_x^\alpha y)}{\partial ({}^C D_x^\alpha y)} = K {}_x D_b^\alpha \frac{1}{(b-x)^{1-\alpha}}. \quad (24)$$

The Euler–Lagrange equation (23) follows from (24) by Theorem 2.2 and the well-known equality  ${}_x D_b^\alpha (b-x)^{\alpha-1} = 0$  (see, e.g., Property 2.1 of [3]). A more important consequence from our optimality condition (19), however, is that we can obtain a new fractional Euler–Lagrange differential equation involving only Caputo derivatives.

**Theorem 4.2** (The Euler–Lagrange equation with only Caputo derivatives). *Consider the problem of extremizing (15) with a Lagrangian  $L \in C^2([a, b] \times \mathbb{R}^2)$  subject to boundary conditions  $y(a) = y_a$  and  $y(b) = y_b$ . If  $y \in C^1([a, b])$  is a solution to this problem, then  $y$  satisfies the fractional Euler–Lagrange differential equation*

$$\frac{\partial L(x, y, {}_a^C D_x^\alpha y)}{\partial y} + {}_x D_b^\alpha \frac{\partial L(x, y, {}_a^C D_x^\alpha y)}{\partial ({}_a^C D_x^\alpha y)} = 0. \quad (25)$$

**Proof.** The optimality condition (25) is obtained taking the right Caputo fractional derivative (7) to both sides of (19). Taking the limit  $x \rightarrow b$  on the left-hand side of (19), and having in mind that  $L \in C^2([a, b] \times \mathbb{R}^2)$  and  $y \in C^1([a, b])$ , we obtain the quantity  $\frac{\partial L(b, y(b), {}_a^C D_b^\alpha y(b))}{\partial ({}_a^C D_b^\alpha y)}$ . On the other hand, the right-hand side of (19) diverges when  $x \rightarrow b$  if  $K \neq 0$ . We conclude that, for  $L \in C^2([a, b] \times \mathbb{R}^2)$ , one has  $K = 0$  and  $\frac{\partial L(b, y(b), {}_a^C D_b^\alpha y(b))}{\partial ({}_a^C D_b^\alpha y)} = 0$  (this is a necessary condition for the Lagrangian  $L$  and the solution  $y$  to be nonsingular at  $x = b$ ). Finally, we obtain (25) by computing the Caputo derivative of the right-hand side of (19) and using Theorem 2.2.  $\square$

From the well-known relations between the Riemann–Liouville and Caputo fractional derivatives (see Remark 2.4), the Euler–Lagrange equation in terms of Caputo fractional derivative (25) can be derived from the Euler–Lagrange equation in terms of Riemann–Liouville fractional derivative (23) by assuming that  $\frac{\partial L(b, y(b), {}_a^C D_b^\alpha y(b))}{\partial ({}_a^C D_b^\alpha y)} = 0$ . Here, we do not assume a priori that  $\frac{\partial L(b, y(b), {}_a^C D_b^\alpha y(b))}{\partial ({}_a^C D_b^\alpha y)} = 0$ , showing it as a consequence of the new fractional Euler–Lagrange equation in integral form (19). The Euler–Lagrange equation (25), involving only Caputo derivatives, is valid for regular boundary conditions. Consequently, equation (25) should be better suited to applications in physics, sciences and engineering than the fractional Euler–Lagrange equation (23) with mixed Riemann–Liouville and Caputo fractional derivatives [6].

## 5 Conclusion

We generalized one of the most important lemmas of the calculus of variations, the DuBois–Reymond fundamental lemma of variational calculus, to functionals depending on fractional derivatives (Lemma 3.2). The new lemma enabled us to prove a fractional Euler–Lagrange equation in integral form containing fractional derivatives of only one type (Theorem 4.1). Furthermore, we also showed that, when the Lagrangian is a  $C^2$  function, one can then obtain a fractional Euler–Lagrange differential equation depending only on Caputo derivatives (Theorem 4.2). This is an important result because differential equations involving only Caputo derivatives are, in general, better suited to applications in physics, sciences and engineering, than a fractional differential equation involving both Caputo and Riemann–Liouville derivatives [6].

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